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# Frequency synchronization in a random oscillator network

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We study the frequency synchronization of a randomly coupled oscillators. By analyzing the continuum limit, we obtain a sufficient condition for the mean-field-type synchronization. We especially find that the critical coupling constant $K$ becomes 0 in the random scale-free network, $P(k) \propto k^{-\gamma}$, if $2 < \gamma \leqslant 3$. Numerical simulations in finite networks are consistent with this analysis.

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## I. INTRODUCTION

Recently, it has become clear that complex networks play an important role in many natural and artificial systems, such as neural networks, metabolic systems, power supply systems, the Internet, and so on [1,2]. In particular, we have recognized that many networks have scale-free topology; the distribution of the degree obeys the power law $P(k) \sim k^{-\gamma}$. The study of scale-free networks now attracts the interest of many researchers in mathematics, physics, engineering, and biology.

The dynamics in network systems is one of the important themes of the investigation of complex networks. In this paper, we study the synchronization of a random network of oscillators. Phase synchronization in complex networks has been studied by several authors [3,4], while frequency synchronization has not been studied as much. One of the important studies on this problem was done by Watts [5]. He suggested, from numerical simulation, that mean-field-type synchronization occurs in small-world networks such as the Watts-Strogatz model. His study was followed by the work of Hong *et al.*, in which phase diagrams and critical exponents are numerically studied in detail [6]. These works showed that mean-field-type synchronization, which Kuramoto observed in globally coupled oscillators [7], appears also in small-world networks. However, such a study in scale-free networks has not been performed yet.

In this paper, we analytically study frequency synchronization in a random network of oscillators. By analyzing the continuum limit of this model, we obtain a sufficient condition for synchronization. Our result shows that in the scale-free random network, the threshold for synchronization is absent if $2 < \gamma \leqslant 3$. We also carry out numerical simulations, and the results are consistent with this analysis.

This paper is constructed as follows. The next section describes the model of an oscillator network and derives the continuum limit equation. Section III is devoted to deriving a sufficient condition for synchronization from the continuum limit equation. We show that the order parameter is different from the one used in previous works; in particular, we conclude that the threshold for synchronization disappears in the random scale-free network. These results are consistent with the results of the numerical simulations, which are described in Sec. IV. In the final section, we make a summary of this paper and discuss the relation to the other properties of the scale-free network.

## II. OSCILLATOR NETWORK MODEL AND ITS CONTINUUM LIMIT

First we describe the model we study in this paper. We study the network with $N$ nodes. At each node, there exists an oscillator, and the phase of the oscillator $\theta_i$ is developed as

$$\frac{\partial \theta_i}{\partial t} = \omega_i + K \sum_j a_{i,j} \sin(\theta_j - \theta_i), \tag{1}$$

where $K$ is the coupling constant, and $a_{i,j}$ is 1 if the nodes $i$ and $j$ are connected and 0 otherwise. $\omega_i$ is a random number whose distribution is given by the function $N(\omega)$.

For the analytic study, it is convenient to use the continuum limit equation. We define $P(k)$ as the distribution of nodes with degree $k$, and $\rho(k,\omega;t,\theta)$ as the density of oscillators with phase $\theta$ at time $t$, for given $\omega$ and $k$. We assume that $\rho(k,\omega;t,\theta)$ is normalized as

$$\int_0^{2\pi} \rho(k,\omega;t,\theta) d\theta = 1. \tag{2}$$

For simplicity, we assume $N(\omega) = N(-\omega)$. Under this assumption, we suppose that the collective oscillation corresponds to the stable solution, $d\rho/dt = 0$, in this model.

Now we construct the continuum limit equation for the network of oscillators. The evolution of $\rho$ is determined by the continuity equation $\partial\rho/\partial t = -\partial(\rho v)/\partial\theta$, where $v$ is defined by the continuum limit of the right-hand side (r.h.s.) of Eq. (1). Because one randomly selected edge connects to the node of degree $k$, frequency $\omega$, and phase $\theta$ with the probability $kP(k)N(\omega)\rho(k,\omega;t,\theta)/\int dk k P(k)$, $\rho(k,\omega;t,\theta)$ obeys the equation

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$$\frac{\partial \rho(k,\omega;t,\theta)}{\partial t} = -\frac{\partial}{\partial \theta}\left[\rho(k,\omega;t,\theta)\left(\omega + \frac{Kk\int d\omega' \int dk' \int d\theta' N(\omega')P(k')k'\rho(k',\omega';t,\theta')\sin(\theta-\theta')}{\int dk' P(k')k'}\right)\right]. \tag{3}$$

In the next section, we study the mean-field solution of this equation.

## III. MEAN-FIELD ANALYSIS OF RANDOM OSCILLATOR NETWORK

In this section, we study the sufficient condition for the synchronization using Eq. (3). First we introduce order parameter $(r,\psi)$ as

$$re^{i\psi} = \int d\omega \int dk \int d\theta N(\omega) \times P(k)k\rho(k,\omega;t,\theta)e^{i\theta} \Big/ \int dkP(k)k. \tag{4}$$

This order parameter is different from the one used in previous work in the small-world model [5,6]. In previous works, $\Sigma_i\, e^{i\theta_i}/N$ is used for the mean field, while our order parameter corresponds to $\Sigma_i\, k_i e^{i\theta_i}/\Sigma_i\, k_i$, where $k_i$ is the degree of the node $i$. However, from Eq. (3) it seems natural to use Eq. (4) as the mean-field value in the random network. Here we note that $0 \leqslant r \leqslant 1$.

Inserting Eq. (4) into Eq. (3), we get

$$\frac{\partial \rho(k,\omega;t,\theta)}{\partial t} = -\frac{\partial}{\partial \theta}\{\rho(k,\omega;t,\theta)[\omega + Kkr\sin(\psi-\theta)]\}. \tag{5}$$

The time-independent solution of $\rho$ is then

$$\frac{\partial}{\partial \theta}\{\rho(k,\omega;t,\theta)[\omega + Kkr\sin(\psi-\theta)]\} = 0. \tag{6}$$

Without a loss of generality, we can assume $\psi=0$. Since we want to seek the solution which corresponds to Kuramoto's solution in globally coupled oscillators, we assume the solution of this equation as

$$\rho(k,\omega;\theta) = \begin{cases} \delta\left(\theta - \arcsin\left(\frac{\omega}{Kkr}\right)\right) & \text{if } \frac{|\omega|}{Kkr} \leqslant 1 \\ \frac{C(k,\omega)}{|\omega - Kkr\sin\theta|} & \text{otherwise,} \end{cases} \tag{7}$$

where $C(k,\omega)$ is the normalization factor. Here we note that $\rho$ depends on both $K$ and $k$. This equation means that $Kk$ corresponds to the coupling between mean field and the oscillator. Inserting Eq. (7) into Eq. (4), we get the equation for $r$,

$$r = \int d\omega \int dk \int d\theta N(\omega)kP(k)\rho(k,\omega;\theta)e^{i\theta} \Big/ \int dkkP(k). \tag{8}$$

To calculate this integral, first we divide the integral over $\omega$,

$$\int d\omega \int dk \int d\theta N(\omega)kP(k)\rho(k,\omega;\theta)e^{i\theta} = \int dk \int d\theta\left(\int_{-Kkr}^{Kkr} d\omega + \int_{-\infty}^{-Kkr} d\omega + \int_{Kkr}^{\infty} d\omega\right) \times N(\omega)kP(k)\rho(k,\omega;\theta)e^{i\theta}. \tag{9}$$

The contribution from the integral at $\omega < -Kkr$ and $\omega > Kkr$ is 0 if $N(\omega)=N(-\omega)$, because

$$\left(\int_{-\infty}^{-Kkr} d\omega + \int_{Kkr}^{\infty} d\omega\right)N(\omega)\rho(k,\omega;\theta)e^{i\theta} = \int_{Kkr}^{\infty} N(\omega)e^{i\theta}C(k,\omega)\left(\frac{1}{\omega - Kkr\sin\theta} + \frac{1}{\omega + Kkr\sin\theta}\right). \tag{10}$$

The integral of the r.h.s. of Eq. (10) over $\theta$ is equal to 0. Therefore, Eq. (9) is equivalent to

$$r = \int dk \int_{-Kkr}^{Kkr} N(\omega)kP(k) \times \exp\left[i\arcsin\left(\frac{\omega}{Kkr}\right)\right] \Big/ \int dkkP(k). \tag{11}$$

If we assume $\arcsin(\omega/Kkr)$ is between $[-\pi/2,\pi/2]$, we get

$$\begin{aligned} r\int dkkP(k) &= \int dk \int_{-Kkr}^{Kkr} d\omega N(\omega)kP(k)\sqrt{1-\left(\frac{\omega}{Kkr}\right)^2} \\ &= \int dk \int_{-1}^{1} d\omega' kP(k)N(Kkr\omega')\sqrt{1-\omega'^2}Kkr \\ &= Kr\int dkk^2P(k)\int_{-1}^{1} d\omega' N(Kkr\omega')\sqrt{1-\omega'^2}. \end{aligned} \tag{12}$$

If $r \neq 0$, we get

$$\int dkkP(k) = K\int dkk^2P(k)\int_{-1}^{1} d\omega' N(Kkr\omega')\sqrt{1-\omega'^2}. \tag{13}$$

The l.h.s. of this equation is independent of $r$ and we define the r.h.s. of this equation as $f(r)$. At $r=1$, $f(r)$ is not larger than $\int dkkP(k)$, because

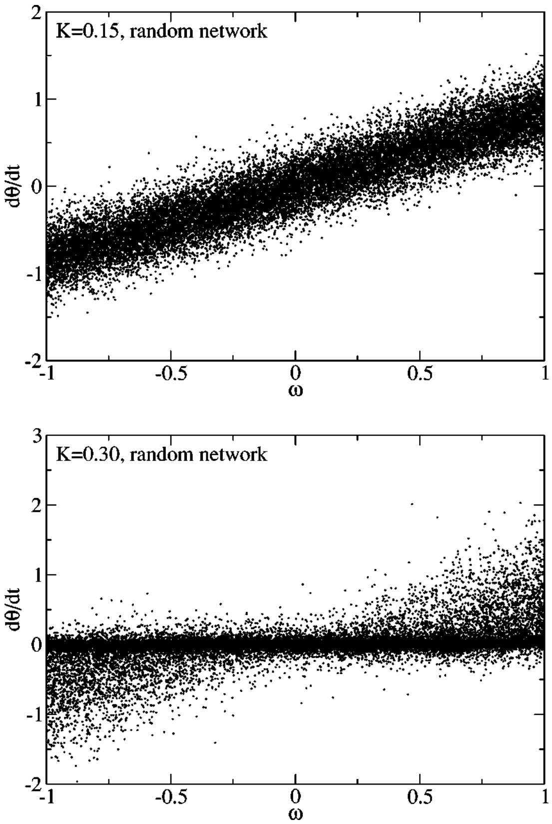


FIG. 1. $(\omega, d\theta/dt)$ distribution of oscillators in a random network, for $K=0.15$ and $K=0.30$.

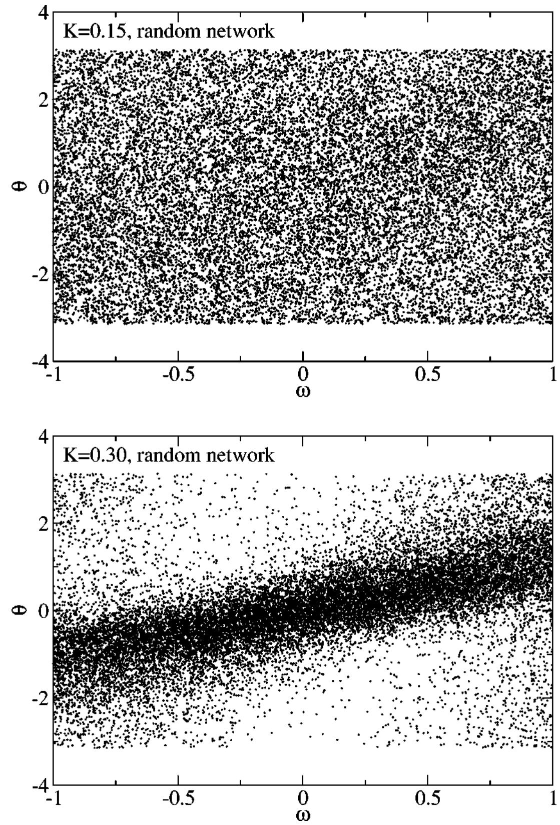


FIG. 2. $(\omega, \theta)$ distribution of oscillators in a random network; $K=0.15$ and $K=0.30$.

$$\int dkk^2P(k)\int_{-1}^{1} d\omega' N(Kkr)\sqrt{1-\omega'^2}$$
$$\leqslant \int dkk^2P(k)\int_{-1}^{1} d\omega' N(Kkr\omega')$$
$$\leqslant \int dkk^2P(k)\frac{1}{Kkr}\int_{-\infty}^{\infty} d\omega'' N(\omega'')$$
$$= \frac{\int dkkP(k)}{Kr}. \qquad (14)$$

Here we use the relation $\int_{-\infty}^{\infty} d\omega N(\omega)=1$. Therefore, the sufficient condition that Eq. (12) have a solution at $0<r\leqslant 1$ is that $f(r)>\int dkkP(k)$ at $r=0$,

$$\frac{KN(0)\pi \int dkk^2P(k)}{2\int dkkP(k)} > 1. \qquad (15)$$

This is the sufficient condition for synchronization in a random network of oscillators. The most impressive point of this equation is that in the random scale-free network, $P(k) \propto k^{-\gamma}$, this condition is satisfied for any $K>0$ if $2<\gamma\leqslant 3$, because $\int dkk^2P(k)/\int dkkP(k)$ diverges. Therefore, we have no threshold for synchronization in the random scale-free network. This seems similar to the absence of a threshold in the susceptible-infected-susceptible (SIS) model [8]. We will discuss this similarity later.

In this section, we derive a sufficient condition for synchronization in a random network of oscillators, using the continuum limit equation. In the next section, we show that the analysis above is in good agreement with the results of the numerical simulations.

## IV. NUMERICAL SIMULATION OF SYNCHRONIZATION

In this section, we show the result of the numerical simulations of the random network of oscillators. In all the simulations, we take $N(\omega)$ as $N(\omega)=0.5$ if $-1.0<\omega<1.0$, and 0 otherwise.

First we show the result on the 1000-node Erdös-Rényi random network model. We choose the probability of coupling $p=0.005$, which gives $\int dkkP(k)=5.0$ and $\int dkk^2P(k)=29.7$ on average. In this case, the estimated critical value $K$ is $K_c=0.214$. Each simulation is carried out 100 times.

In Fig. 1, we plot the relation between $\omega_i$ and $d\theta_i/dt$ after a long time $(t=200)$ when $K=0.15$ and 0.30. In the case of $K=0.15$, $d\theta/dt$ seems to depend on $\omega$ linearly. On the other hand, at $K=0.30$ many oscillators seem to be synchronized at $d\theta/dt=0$. This figure strongly suggests that synchronization occurs between $K=0.15$ and 0.30.

We plot the relation between $\omega$ and $\theta$ for $K=0.15$ and $K=0.30$ in Fig. 2. We find a clear difference between these two cases. In the case of $K=0.30$, the distribution of $(\omega_i, \theta_i)$ is apparently nonuniform, while at $K=0.15$ we cannot find any structure. In the case of $K=0.30$, $\theta$ seems to depend linearly on $\omega$. However, from the previous analysis we suggest that $\theta$ depends on both $\omega$ and $k$. To clarify the degree dependence, we plot $(\omega, \theta)$ for the nodes with the degree $k$= 3, 5, and 7 at

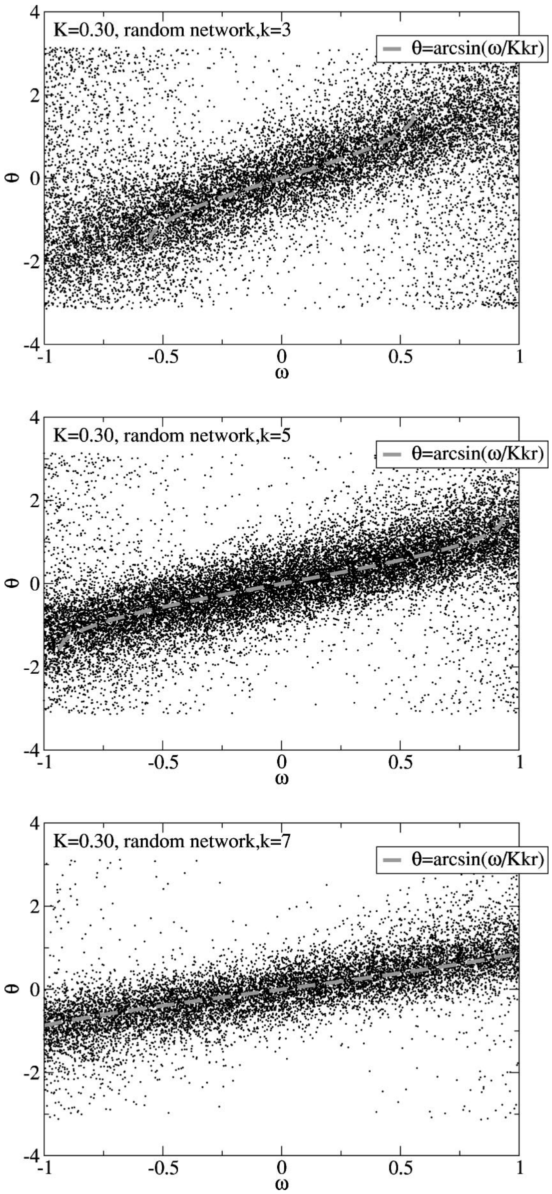


FIG. 3. $(\omega,\theta)$ distribution of oscillators with degree 3, 5, and 7 in a random network; $K=0.30$.

$K=0.30$ in Fig. 3. We also plot $\arcsin(\omega/Kkr)$ in these figures. The average of $r$ is 0.623 in our simulation. From these figures, we find that distribution of $(\omega,\theta)$ seems to be concentrated around a single line. The concentration line coincides qualitatively with $\theta=\arcsin(\omega/Kkr)$. This result suggests that our mean field defined by Eq. (4) is the correct one.

To estimate the critical coupling $K_c$, we plot the $K$ dependence of the average of the order parameter $r_{\mathrm{av}}$ in Fig. 4. $r_{\mathrm{av}}$ is less than 0.1 and shows a weak dependence on $K$ at $K<0.2$. This nonzero value of $r_{\mathrm{av}}$ is due to the finite-size effect. On the other hand, at $K>0.2$, $r_{\mathrm{av}}$ increases rapidly as the interaction increases. This figure suggests that $K_c$ is about 0.2, which is in agreement with our analysis. Therefore, we conclude that all numerical results are consistent with our analysis.

From these simulations, we find that our mean-field theory is applicable to the Erdös-Rényi model. However, the

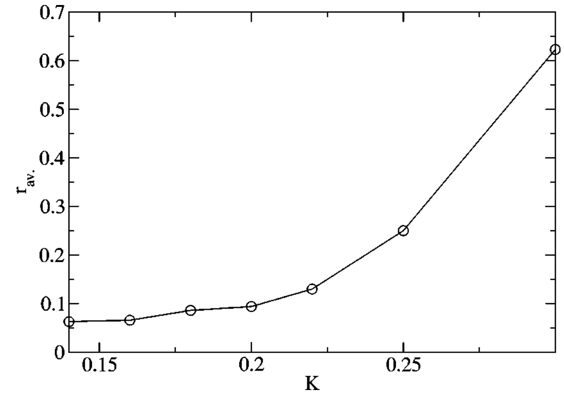


FIG. 4. Interaction dependence of mean-field parameter $r_{\mathrm{av}}$.

most impressive suggestion of our analysis is the absence of the threshold in the random scale-free network. In the following, we show the result of simulation in the random scale-free network with $\gamma=2.5$. In Fig. 5, we show the relation between order parameter $r$ and coupling constant $K$ for $N=500$, 1000, 2000, and 4000. At $N=500$, $r_{\mathrm{av}}$ rapidly increases above $K\sim0.16$, which is qualitatively consistent with the $K_c\sim0.175$ estimated from Eq. (12). As the network size increases, $r_{\mathrm{av}}$ at small coupling decreases, which suggests that the finite $r_{\mathrm{av}}$ at small coupling is the finite-size effect. The order parameter begins to increase rapidly above $K_c$. We note that by increasing the system size, the increase of order parameter begins at smaller coupling. This means that the critical coupling $K_c$ decreases as the system size increases. We also show $K_c$ estimated from Eq. (12) in this figure. The estimated $K_c$ qualitatively coincides with the coupling constant at which the order parameter increases rapidly. We conclude that our analysis and the results of the numerical simulation show a good agreement also in the random scale-free network. These results suggest that in the infinite-size scale-free network, the critical coupling constant $K_c$ becomes zero, just the same as in the continuum limit equation.

To compare the results of the numerical simulation and the analysis more precisely, we need a more accurate estimation of $K_c$ from the numerical simulation. In the case of the globally coupled networks and Watts-Strogatz model, $K_c$ is

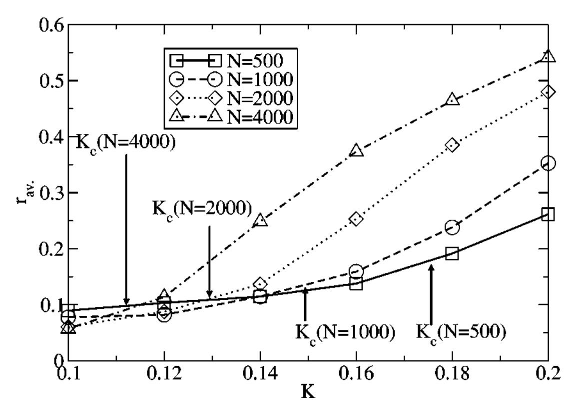


FIG. 5. Interaction dependence of the mean-field parameter in the random scale-free network for $N=500$, 1000, 2000, and 4000. The arrow shows $K_c$ estimated from Eq. (12). Simulations for each parameter are carried out for at least 50 realizations of the networks.

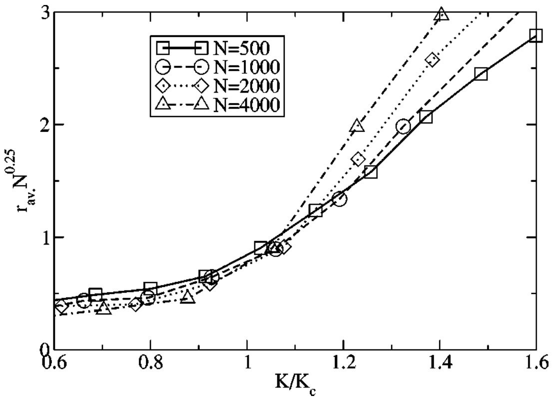


FIG. 6. The relation between $N^{0.25}r_{\rm av}$ and $K/K_c$, where $K_c$ is the value estimated from Eq. (12).

numerically obtained as the point at which $N^{0.25}r_{\rm av}$ becomes independent of the size of the network [6]. In their analysis, there exists an assumption that $K_c$ does not depend on the size of the network. On the other hand, our analysis and simulation show that $K_c$ depends clearly on the size of the network through the average of the square of the degrees. Therefore, it is impossible to obtain accurate $K_c$ from finite-size analysis. The exact estimation of $K_c$ is a difficult task.

However, we find that $K_c$ derived from Eq. (12) seems to have a strong relation to the phase transition. We rescale $K$ by $K_c$, which is obtained from Eq. (12), and plot the relation between $N^{0.25}r_{\rm av}$ and $K/K_c$ in Fig. 6. In the case of $N=1000$, 2000, and 4000, the well-defined crossing point exists at $K/K_c=1$. In the case of $N=500$, $N^{0.25}r_{\rm av}$ at $K/K_c=1$ is a little larger than in the other cases. However, this difference is small, and it seems that $K/K_c=1.0$ is the crossing point at large $N$. This result is similar to the results of the finite-size scaling in the globally coupled networks and Watts-Strogatz model. In these models, there exists a crossing point at $K=K_c$. On the other hand, our analysis is not a precise determination of the critical coupling strength. To avoid the size dependence of $K_c$, we rescale $K$ and we have no guarantee that such a rescaling is valid for the scale-free network model. However, our result strongly suggests that $K_c$ obtained from the numerical simulation coincides with the result of the analytic solution.

To conclude this section, we carried out the simulations on the Erdös-Rényi model and the random scale-free network. All the results of these simulations show a qualitative agreement with the analysis in the previous section.

## V. SUMMARY AND DISCUSSION

In this paper, we study the frequency synchronization of the random oscillator network. By analyzing the continuum limit equation, we find that mean-field-type synchronization occurs in a random network model. We obtain a sufficient condition for the synchronization. In particular, we find that the threshold for the synchronization is absent in a scale-free random network if $2<\gamma\leqslant 3$. The results of numerical simulations are in good agreement with this analysis.

One of the most astonishing results in the dynamics of the scale-free network is the absence of an epidemic threshold in the SIS model. Our result seems to be similar to the result in the SIS model, however there is a large difference between them. In the SIS model, the absence of an epidemic threshold is the result of the divergence of $k_{\rm nn}$, the mean degree of the nearest-neighbor nodes [9]. On the other hand, in our model the absence of a threshold originates from the degree dependence of the coupling between the order parameter and the oscillators. The coupling between the oscillators and the mean field is proportional to the degree of the nodes, as shown in Eq. (5), and the contribution to the order parameter from the oscillator is also proportional to the degree of the node, shown in Eq. (4). This degree dependence results in the $k^2$ dependence of Eq. (15), which leads to the absence of the threshold in a random scale-free network. Therefore, there is a large difference between the absence of a threshold in the SIS model and the synchronization, although these are apparently similar results. To clarify this difference, we will need to study the synchronization in the other network models. As Eguíluz and Klemm have shown, a scale-free network with a large clustering coefficient has an epidemic threshold in the SIS model [10], due to the smallness of $k_{\rm nn}$. The different behavior of the threshold may appear in our oscillator network, because the absence of the threshold is not caused by the divergence of $k_{\rm nn}$, but by the degree dependence of mean-field–oscillator coupling. The study of the synchronization in other scale-free network models is a future problem.

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